ON AN UNSYMMETRICAL PROBABILITY CURVE.

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WHEN repeated observations of a quantity are made, and are liable to error through accidental or unknown causes, it is sometimes found that the facility of error is greater on one side of the arithmetical mean than on the other, so that the limits of possible error are different, and + and — errors of equal amount do not occur with equal frequency. Cases of this kind have been noticed by various writers; see for instance Quetelet's *Lettres sur la The'orie des Probabilite's*, pp. 180 and 410. He remarked a similarity between the apparent form of the curve of facility, and that of the series of terms in the expansion of a binomial $p+q$ to a high but finite power, when p and q are very unequal. I am not aware that any writer has attempted to give the analytical equation of such a curve, and it is the object of the present paper to obtain one. We shall do this, however, in the most general way, regarding the desired curve of facility as a limiting form of the series of coefficients in the expansion of a polynomial, which may or may not be a binomial. The limit of a polynomial having none but positive coefficients has been investigated in a peculiar manner, by Laplace and subsequent writers, and found to be the common probability curve. See, for instance, Meyer, *Wahrscheinlichkeitsrechnung*, Leipsic 1879, pp. 141, 350, 407, 442.* A simpler and very different way of obtaining such results was given by me in the ANALYST, Sept. 1879 and Sept. 1881, and I have extended it to the cases of polynomials of two and three variables. Although the general form of the ultimate limiting curve for a polynomial of one variable is represented by the probability curve

$$y = \frac{hdx}{\sqrt{\pi}} e^{-h^2x^2}, \tag{1}$$

yet it can be shown that the actual form of an expansion to a high power approaches still more closely to another and more complex curve, of which (1) is only a special case. To discover the nature of this curve, we must take precautions to insure the most accurate approximation.

First, we shall use the method of symmetrical differences. Having an unlimited series of equidistant terms

$$\ldots .\ l_{i-2},\ l_{i-1},\ l_i,\ l_{i+1},\ l_{i+2},\ \ldots . \tag{2}$$

we take their differences so as to keep l_i always in the middle. The differences of even order are

*Meyer's work is a valuable compendium of the science of probability, brought down to recent date.

$$\varDelta_2 = -2l_i+l_{i+1}+l_{i-1},\quad \varDelta_4 = 6l_i-4(l_{i+1}+l_{i-1})+l_{i+2}+l_{i-2},$$
$$\varDelta_6 = -20l_i+15(l_{i+1}+l_{i-1})-6(l_{i+2}+l_{i-2})+l_{i+3}+l_{i-3},\ \text{\&c., \&c.}$$

Those of odd order would in the usual form be

$$\varDelta_1 = l_{i+½}-l_{i-½},\qquad \varDelta_3 = l_{i+1½}-3l_{i+½}+3l_{i-½}-l_{i-1½},\ \text{\&c.}$$

But since only the terms in (2) are supposed to be given, we make the hypotheses

$$l_{i-½} = \tfrac{1}{2}(l_{i-1}+l_i),\quad l_{i+½} = \tfrac{1}{2}(l_i+l_{i+1}),\quad l_{i+1½} = \tfrac{1}{2}(l_{i+1}+l_{i+2}),\ \text{\&c.},$$

and by substitution get expressions which we agree to represent by $\varDelta_1$, $\varDelta_3$, &c., thus

$$\left.\begin{aligned}\varDelta_1 &= \tfrac{1}{2}(l_{i+1}-l_{i-1}),\quad \varDelta_3 = \tfrac{1}{2}[l_{i+2}-l_{i-2}-2(l_{i+1}-l_{i-1})],\\ \varDelta_5 &= \tfrac{1}{2}[l_{i+3}-l_{i-3}-4(l_{i+2}-l_{i-2})+5(l_{i+1}-l_{i-1})],\ \text{\&c.}\end{aligned}\right\}\quad (3)$$

It will be found that each difference of odd order thus formed is half the sum of the two nearest differences of the same order formed in the usual way, one on each side of the centre or place of l_i. Thus for example

$$\varDelta_3 = \tfrac{1}{2}[(l_{i+2}-3l_{i+1}+3l_i-l_{i-1})+(l_{i+1}-3l_i+3l_{i-1}-l_{i-2})].$$

From the expressions for $\varDelta_1$, $\varDelta_2$, &c. we get by successive eliminations

$$l_{i\pm1} = l_i\pm\varDelta_1+\tfrac{1}{2}\varDelta_2,\qquad l_{i\pm2} = l_i\pm2\varDelta_1+2\varDelta_2\pm\varDelta_3+\tfrac{1}{2}\varDelta_4,$$

and in general

$$l_{i+n} = l_i+\frac{n}{1}\varDelta_1+\frac{n^2}{1.2}\varDelta_2+\frac{n(n^2-1^2)}{1.2.3}\varDelta_3+\frac{n^2(n^2-1^2)}{1.2.3.4}\varDelta_4+\frac{n(n^2-1^2)(n^2-2^2)}{1.2.3.4.5}\varDelta_5 + \text{\&c.},\quad (4)$$

where n may be either positive or negative. This formula, which may be used for making interpolations, is given by Lacroix, *Calcul Diff. et Int.*, Paris 1819, Vol. III. pp. 26–28.

It was shown in my ANALYST articles above cited, that if any polynomial

$$\lambda_{-m}z^{-m}+\ldots\ldots+\lambda_{-1}z^{-1}+\lambda_0+\lambda_1z+\ldots\ldots+\lambda_mz^m,\quad (5)$$

whose coefficients λ may be either $+$ or $-$, is expanded to the k power, and the expansion is written

$$l_{-km}z^{-km}+\ldots+l_{-1}z^{-1}+l_0+l_1z+\ldots+l_{km}z^{km},\quad (6)$$

then any coefficient l_i in the expansion, and the $2m$ coefficients nearest to it, will be connected by the relation

$$\frac{(\lambda_1l_{i-1}-\lambda_{-1}l_{i+1})+2(\lambda_2l_{i-2}-\lambda_{-2}l_{i+2})+\ldots+m(\lambda_ml_{i-m}-\lambda_{-m}l_{i+m})}{\lambda_0l_i+(\lambda_1l_{i-1}+\lambda_{-1}l_{i+1})+(\lambda_2l_{i-2}+\lambda_{-2}l_{i+2}+\ldots+(\lambda_ml_{i-m}+\lambda_{-m}l_{i+m})} = \frac{i}{k+1}.\quad (7)$$

Let l_{i+1}, l_{i-1} &c, be expressed in terms of l_i and the differences as in (4), and write also

$$\left.\begin{aligned}b_0 &= \lambda_0+(\lambda_1+\lambda_{-1})+(\lambda_2+\lambda_{-2})+\ldots\ldots+(\lambda_m+\lambda_{-m})\\ b_1 &= 1(\lambda_1-\lambda_{-1})+2(\lambda_2-\lambda_{-2})+\ldots\ldots+m(\lambda_m-\lambda_{-m})\\ b_2 &= 1^2(\lambda_1+\lambda_{-1})+2^2(\lambda_2+\lambda_{-2})+\ldots\ldots+m^2(\lambda_m+\lambda_{-m})\\ b_3 &= 1^3(\lambda_1-\lambda_{-1})+2^3(\lambda_2-\lambda_{-2})+\ldots\ldots+m^3(\lambda_m-\lambda_{-m})\\ &\qquad\text{\&c.}\qquad\qquad\qquad\qquad \text{\&c.}\end{aligned}\right\}\quad (8)$$

Denoting the numerator and denominator in the first member of (7) by N and D, and writing $1.2.3 \ldots n = n\,!$, we get

$$\left.\begin{aligned}
N &= b_1 l_i - b_2 \Delta_1 + \tfrac{1}{2} b_3 \Delta_2 - \frac{1}{3\,!}(b_4 - b_2)\Delta_3 + \frac{1}{4\,!}(b_5 - b_3)\Delta_4 - \frac{1}{5\,!} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times (b_6 - 5b_4 + 4b_2)\Delta_5 \\
&\quad + \frac{1}{6\,!}(b_7 - 5b_5 + 4b_3)\Delta_6 - \frac{1}{7\,!}(b_8 - 8b_6 + 19b_4 - 12b_2)\Delta_7 + \&c. \\
D &= b_0 l_i - b_1 \Delta_1 + \tfrac{1}{2} b_2 \Delta_2 - \frac{1}{3\,!}(b_3 - b_1)\Delta_3 + \frac{1}{4\,!}(b_4 - b_2)\Delta_4 - \frac{1}{5\,!} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times (b_5 - 5b_3 + 4b_1)\Delta_5 \\
&\quad + \frac{1}{6\,!}(b_6 - 5b_4 + 4b_2)\Delta_6 - \frac{1}{7\,!}(b_7 - 8b_5 + 19b_3 - 12b_1)\Delta_7 + \&c, \\
\frac{N}{D} &= \frac{i}{k+1}.
\end{aligned}\right\} \quad (9)$$

It will be seen that in the coefficient of Δ_{2m+1} or Δ_{2m+2}, the numerical coefficients of the b within the parentheses are those of the powers of n in the product of the factors

$$(n^2 - 1^2)\,(n^2 - 2^2) \ldots . (n^2 - m^2).$$

When k becomes an infinity of the second order, that is, of a magnitude comparable with the quotient of a finite area by $(dx)^2$, and the successive values of l are regarded as consecutive ordinates y to a limiting curve which extends to an infinite distance over the axis of X, we have

$$l_i = y, \qquad \Delta_1 = dy, \qquad \Delta_2 = d^2y, \qquad \Delta_3 = d^3y, \ \&c.$$

The common interval Δx between ordinates becomes dx when they are set close together, and the abscissa corresponding to any y is $x = idx$. Thus (9) becomes the differential equation of the curve, and b_0, b_1, &c. are constants. Any given polynomial may be reduced to one in which $\Sigma(\lambda) = 1$, by dividing it throughout by the sum of its coefficients. We have then $b_0 = 1$. If a constant number is added to or subtracted from all the exponents of z in (5), it will not alter the values of l in (6). Hence, as shown in my former article, b_1 may be reduced to zero by transferring the origin or place of z^0 to the centre of parallel forces of the coefficients λ in (5), when these are regarded as forces ranged equidistantly along the imponderable axis of X and acting upon it at right angles in the plane XY. Then any constant b_n in (8) will denote the sum of the products formed by multiplying each λ into the nth power of its abscissa reckoned from the new origin, if the common interval Δx between the coefficients or forces λ is regarded as unity. But if any other unit of abscissas is employed, the sum of the products will be $b_n(\Delta x)^n$, or $b_n(dx)^n$ when the coefficients are set close together so as to be consecutive. We may now write (9) as follows.

$$\frac{b_2dy-\frac{1}{2}b_3d^2y+\frac{1}{6}(b_4-b_2)d^3y-\&c}{y+\frac{1}{2}b_2d^2y-\frac{1}{6}b_3d^3y+\&c.}=\frac{-x}{(k+1)dx}. \tag{10}$$

In the numerator of the first member let d^3y, d^4y &c. be neglected in comparison with dy and d^2y, and in the denominator neglect d^2y, d^3y &c. in comparison with y. Since k is infinitely large, we may write k instead of $k+1$. Therefore

$$\frac{dy-\frac{1}{2}(b_3\div b_2)d^2y}{y}=\frac{-x}{kb_2dx}.$$

Invert both members of this equation, subtract $\frac{1}{2}(b_3\div b_2)$ from each, and invert them both back again. This gives

$$\frac{dy-\frac{1}{2}(b_3\div b_2)d^2y}{y-\frac{1}{2}(b_3\div b_2)dy+\frac{1}{4}(b_3\div b_2)^2d^2y}=\frac{-x}{kb_2dx+\frac{1}{2}(b_3\div b_2)x}. \tag{11}$$

In the denominator of the first member, let d^2y be neglected in comparison with y and dy. From the properties of the lever arms of the coeffic'nts in polynomials and their products, as shown by me in the ANALYST articles cited (see also March and Nov. 1880), it follows that since the origin or place of z^0 in the given polynomial (5) is transferred to the centre of forces for the coefficients λ, the origin or place of z^0 in the expansion to the k power will be located at the centre of forces for the coefficients l in (6), which become the ordinates y to the limiting curve.

When the λ's are all positive, as they must be if they represent probabilities, the y's will be all positive, and their centre of forces is the same as their centre of gravity, if the ordinates are regarded as the masses of material points ranged along the X axis at intervals equal to dx. Now in (11) let the origin be transferred from the centre of gravity to another convenient point by putting

$$x-\frac{2kb_2^2\,dx}{b^3} \tag{12}$$

in place of x. This gives

$$\frac{dy-\frac{1}{2}(b_3\div b_2)d^2y}{y-\frac{1}{2}(b_3\div b_2)dy}=\frac{4kb_2dx-2(b_3\div b_2)x}{(b_3\div b_2)^2x}. \tag{13}$$

In the first member, the numerator is the differential of the denominator. Without any further change of origin, we can write approximately

$$y+\frac{1}{2}(b_3\div b_2)dy \qquad \text{and} \qquad x+\frac{1}{2}(b_3\div b_2)dx$$

in place of y and x respectively, neglecting d^3y in the numerator and d^2y in the denominator, and so get

$$\frac{dy}{y}=\frac{4kb_2dx-(b_3\div b_2)^2dx-2(b_3\div b_2)x}{(b_3\div b_2)^2[x+\frac{1}{2}(b_3\div b_2)dx]}.$$

Since the denominator y in the first member is supposed to be infinitely greater than the numerator dy, the denominator in the second member must

be infinitely greater than its numerator, so that in the denominator we may neglect dx in comparison with x. Also let the constants be expressed by means of two new constants

$$a = \frac{2b_2(dx)^2}{b_3(dx)^3}, \qquad b = kb_2(dx)^2. \tag{14}$$

Since k is supposed to be an infinity of the second order, b represents a finite area. The equation will now stand

$$\frac{dy}{y} = \frac{dx}{x}(a^2b - 1) - a\,dx, \tag{15}$$

and integration gives

$$\log' y = (a^2b - 1)\log' x - ax + \log' C,$$
$$\therefore y = Cx^{a^2b-1}e^{-ax}. \tag{16}$$

This is a more accurate relation between y and x than we could have obtained without the use of symmetrical differences. In (15) dy is not the difference of the ordinates at x and $x+dx$, but of those at $x - \frac{1}{2}dx$ and $x + \frac{1}{2}dx$.

The finite difference of $\log' y$ is usually considered to be

$$\left.\begin{aligned} \Delta \log' y &= \log'(y+\Delta y) - \log' y = \log'\left(1+\frac{\Delta y}{y}\right) \\ &= \frac{\Delta y}{y} - \tfrac{1}{2}\left(\frac{\Delta y}{y}\right)^2 + \tfrac{1}{3}\left(\frac{\Delta y}{y}\right)^3 - \&c. \end{aligned}\right\} \tag{17}$$

But we have taken it to be

$$\left.\begin{aligned} \Delta \log' y &= \log'(y+\tfrac{1}{2}\Delta y) - \log'(y-\tfrac{1}{2}\Delta y) = \log'\left(1+\frac{\Delta y}{2y}\right) - \log'\left(1-\frac{\Delta y}{2y}\right) \\ &= \frac{\Delta y}{y} + \frac{1}{3.2^2}\left(\frac{\Delta y}{y}\right)^3 + \frac{1}{5.2^4}\left(\frac{\Delta y}{y}\right)^5 + \&c. \end{aligned}\right\} \tag{18}$$

These expressions show that when $dy \div y$ is regarded as the differential of $\log' y$, the magnitude of the error committed is either

$$\frac{1}{2}\left(\frac{dy}{y}\right)^2 \qquad \text{or} \qquad \frac{1}{12}\left(\frac{dy}{y}\right)^3, \tag{19}$$

according as we consider dy to be $y_{i+1} - y_i$ or $y_{i+½} - y_{i-½}$. The second error is of a lower order of magnitude than the first. Conversely, if we take $\log' y$ to be the integral of $dy \div y$, the accuracy of the result is increased when dy or Δy is understood in the sense here adopted.

When we write

$$ax = v, \qquad a^2b = n, \tag{20}$$

the ordinate y in (16) is seen to be proportional to an expression having the known form

$$v^{n-1}e^{-v},$$

the element-function of the Gamma integral. The limiting form (16) of the expansion of a polynomial, therefore, is a curve shaped like the one whose area, between $x = 0$ and $x = \infty$, is $\Gamma(n)$. For convenience, we may call it the gamma curve. It makes $y = 0$ for $x = 0$ and for $x = \infty$. (See Price's *Calculus*, I. p. 208.) We also have

$$\left.\begin{aligned} \frac{dy}{dx} &= Cx^{a^2b-2}\, e^{-ax}(a^2b-1-ax), \\ \frac{d^2y}{dx^2} &= Cx^{a^2b-3}\, e^{-ax}\left[(ax)^2-2(a^2b-1)ax+(a^2b-1)(a^2b-2)\right], \end{aligned}\right\} \tag{21}$$

whence it appears that the X axis is tangent to the curve at $x = 0$ and asymptote to it at $x = \infty$, and that y is a maximum at

$$x = ab - \frac{1}{a}. \tag{22}$$

There are two points of inflexion, equidistant from the maximum, at

$$x = ab - \frac{1}{a} \pm \frac{1}{a}\sqrt{(a^2b - 1)}. \tag{23}$$

Since the first $n—2$ differential coefficients will contain both x and e^{-x} as factors, it appears that the X axis has a contact of an order as high as the $n - 2$, with the curve at $x = 0$ and $x = \infty$. Hence the curve anywhere near those points is almost a straight line coinciding with the axis, and values of y in the vicinity of either will scarcely differ from zero. This agrees with what we know of the extreme smallness of the coefficients at or near the two ends of the expansion of a binomial or polynomial to a high power.

Since $\Sigma(\lambda) = 1$ in the given polynomial and $\Sigma(l) = 1$ in its expansion, we shall have $\Sigma(y) = 1$ in the curve (16), so that

$$\frac{1}{dx}\int_0^\infty y\,dx = 1, \qquad \therefore \frac{C}{a^{a^2b}\,dx}\int_0^\infty (ax)^{a^2b-1}\, e^{-ax}d(ax) = 1, \tag{24}$$

which gives the value of C, and we get

$$y = \frac{a\,dx}{\Gamma(a^2b)}(ax)^{a^2b-1}\, e^{-ax}, \tag{25}$$

the complete equation of the curve sought. According to the view stated in my former articles, by which the limiting form of the expansion of a polynomial expresses the most plausible law of facility of error, y will represent the probability that any error which occurs will fall within an arbitrary but very small interval dx, the abscissa of whose middle point is x. The constants a and b depend on the nature of the observations. The quantity $b_2(dx)^2$ is the square of what I have called the "radius of gyration" of the coefficients λ in the polynomial (5), about their centre of gravity; but it might better, perhaps, be called the *quadratic radius*.

I have shown (ANALYST, Jan. and May 1880), that the squared radius for the expansion to the k power is k times as great as for the first power. It follows that b in (14) is the square of the quadratic mean error, that is, the mean of the squares of the deviations of the observations from their arithmetical mean. The expression for a in (14) has for its denominator the quantity $b_3(dx)^3$, which is of the same nature as $b_2(dx)^2$, except that the cubes of the deviations are used instead of the squares, deviations on one side being regarded as + and on the other side as —. We will now prove by a precisely analogous method, that this quantity also is k times greater in the expansion to the k power, than it is in the first power.

In my article last cited, any two polynomial factors were denoted by

$$\left.\begin{aligned} &a_0+a_1z+a_2z^2+\ldots\ldots+a_mz^m, \\ &c_0+c_1z+c_2z^2+\ldots\ldots+c_nz^n. \end{aligned}\right\} \qquad (26)$$

The coefficients a nnd c may be essentially either + or —. Their sums were

$$S_1 = a_0+a_1+\ldots\ldots+a_m, \qquad S_2 = c_0+c_1+\ldots\ldots+c_n, \qquad (27)$$

and h_1 and h_2 were the lever arms of the coefficients about the place of the first terms, so that

$$S_1h_1=(a_1+2a_2+\ldots+ma_m)\Delta x, \quad S_2h_2=(c_1+2c_2+\ldots+nc_n)\Delta x. \quad (28)$$

It was proved that the lever arm of the coefficients in the product of the two factors, about the place of the first term in the product, is

$$H = h_1 + h_2. \qquad (29)$$

The quadratic radii of the coefficients in the two factors, about the places of the first terms, were denoted by g_1 and g_2, whence

$$\left.\begin{aligned} S_1g_1^2 &= (1^2a_1+2^2a_2+\ldots+m^2a_m)(\Delta x)^2, \\ S_2g_2^2 &= (1^2c_1+2^2c_2+\ldots+n^2c_n)(\Delta x)^2, \end{aligned}\right\} \qquad (30)$$

and putting G for the like radius in the product, it was proved that

$$G^2 = g_1^2 + g_2^2 + 2h_1h_2. \qquad (31)$$

Now let u_1^3 and u_2^3 be formed from the two factors in the same way as $b_3(dx)^3$ was. $S_1u_1^3$ and $S_2u_2^3$ will represent the two sums of all the products obtained by multiplying each coefficient by the cube of its + or — deviation from the centre of forces, so that

$$\left.\begin{aligned} S_1u_1^3 &= a_0(-h_1)^3+a_1(\Delta x-h_1)^3+a_2(2\Delta x-h_1)^3\ldots+a_m(m\Delta x-h_1)^3, \\ S_2u_2^3 &= c_0(-h_2)^3+c_1(\Delta x-h_2)^3+c_2(2\Delta x-h_2)^3\ldots+c_n(n\Delta x-h_2)^3. \end{aligned}\right\} \quad (32)$$

For convenience we will call such quantities as u_1 and u_2 *cubic radii* about the point from which the deviations are reckoned. Let l_1 and l_2 denote the cubic radii taken about the first term in each polynomial, then

$$\left.\begin{aligned} S_1l_1^3 &= (1^3a_1+2^3a_2+\ldots+m^3a_m)(\Delta x)^3, \\ S_2l_2^3 &= (1^3c_1+2^3c_2+\ldots+n^3c_n)(\Delta x)^3. \end{aligned}\right\} \qquad (33)$$

From (32) we have

$$S_1 u_1^3 = (1^3a_1+2^3a_2+\,..\,+m^3a_m)(\varDelta x)^3 - 3h_1(1^2a_1+2^2a_2+\,..\,+m^2a_m)(\varDelta x)^2$$
$$+3h_1^2(a_1+2a_2+\,..\,+ma_m)\varDelta x - h_1^3(a_0+a_1+\,..\,+a_m),$$

and a similar expression for $S_2u_2^3$. Then by help of (27), (28), (30) and and (33), we get

$$u_1^3 = l_1{}^3 - 3g_1^2h_1+2h_1^3, \qquad u_2^3 = l_2{}^3 - 3g_2^2h_2+2h_2^3, \tag{34}$$

Likewise denoting by U and L the cubic radii for the product, about its centre of forces and its first term respectively, we shall have

$$U^3 = L^3 - 3G^2H + 2H^3. \tag{35}$$

The sum of all the coefficients in the product is S_1S_2. Supposing the first polynomial factor to be multiplied successively by the terms of the second one, L^3 will be expressed thus,

$$S_1S_2L^3 = c_0(1^3a_1+2^3a_2+\,...\,+m^3a_m)(\varDelta x)^3$$
$$+\,c_1[1^3a_0+2^3a_1+\,...\,+(m+1)^3a_m](\varDelta x)^3$$
$$+\,c_2[2^3a_0+3^3a_1+\,...\,+(m+2)^3a_m](\varDelta x)^3$$
$$+\,...\,+c_n[n^3a_0+(n+1)^3a_1+\,...\,+(n+m)^3a_m](\varDelta x)^3.$$

The coefficient of $c_n(\varDelta x)^3$ is reducible by means of (27), (28), (30) and (33) to

$$S_1\left\{ n^3+3n^2\left(\frac{h_1}{\varDelta x}\right)+3n\left(\frac{g_1}{\varDelta x}\right)^2+\left(\frac{l_1}{\varDelta x}\right)^3 \right\}.$$

Assigning to n the values 0, 1, 2 &c. in succession, we get expressions for the coefficients of $c_0(\varDelta x)^3$, $c_1(\varDelta x)^3$, &c., and so find

$$S_1S_2L^3 = c_0S_1l_1{}^3+c_1S_1[1^3(\varDelta x)^3+3.1^2h_1(\varDelta x)^2+3.1g_1^2\varDelta x+l_1{}^3]$$
$$+c_2S_1[2^3(\varDelta x)^3+3.2^2h_1(\varDelta x)^2+3.2g_1^2\varDelta x+l_1{}^3]$$
$$+\,...\,+c_nS_1[n^3(\varDelta x)^3+3n^2h_1(\varDelta x)^2+3ng_1^2\varDelta x+l_1{}^3],$$

which we put in the form

$$S_2L^3 = l_1{}^3(c_0+c_1+\,...\,+c_n)+3g_1^2\varDelta x(c_1+2c_2+\,...\,+nc_n)$$
$$+\,3h_1(\varDelta x)^2(1^2c_1+2^2c_2+\,...\,+n^2c_n)+(\varDelta x)^3(1^3c_1+2^3c_2+\,...$$
$$..\,+n^3c_n).$$

By help of (27), (28), (30) and (33) this is reduced to

$$L^3 = l_1^3 + l_2^3 + 3g_1^2h_2 + 3g_2^2h_1. \tag{36}$$

Substituting in (35) the values of H, G and L from (29), (31) and (36), we get

$$U^3 = l_1^3+l_2^3 - 3(g_1^2h_1+g_2^2h_2)+2(h_1^3+h_2^3),$$

and by (34) we have finally

$$U^3 = u_1^3 + u_2^3. \tag{37}$$

[To be continued.]